Liu’s Generalized Intuitionistic Fuzzy Sets

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Abstract

In this paper, an extensional generalized intuitionistic fuzzy set, called Liu’s generalized intuitionistic fuzzy set, is proposed. It is showed that not only Atanassov’s intuitionistic fuzzy set but also Mondal & Samanta’s generalized intuitionistic fuzzy set is a special case of this new one. Responding the generalized intuitionistic fuzzy set, the generalized intuitionistic fuzzy value (*GIFV*) is also proposed. Some comparable examples are given, and some important notions and basic algebraic properties of this new generalized intuitionistic fuzzy set and its generalized intuitionistic fuzzy value are discussed.

Keywords: intuitionistic fuzzy set, generalized intuitionistic fuzzy set, generalized intuitionistic fuzzy value, order-preserving order, addition-invariant order

I、Introduction

Intuitionistic fuzzy set (*IFS*) was proposed by Atanassov (1986, 1999) characterized by a membership function and a non-membership function, which is a generalization of Zadeh's fuzzy set (Zadeh, 1965), whose basic component is only a membership function. Over the last decades, *IFS* has been applied to many different fields, such as decision making, logic programming, topology, medical diagnosis, pattern recognition, machine learning and market prediction, etc. (Tan & Chen, 2010; Xu, 2007)

A generalized intuionistic fuzzy set (*GIFS*) with some algebraic properties were proposed by Mondal and Samanta (2002), unfortunately, the addition operation and scalar multiplication operation of Atanassov's *IFS* can no more be used for Mondal and Samanta's *GIFS*, and up to now, their *GIFS* do not have its own addition operation and scalar multiplication operation like which of Atanassov's *IFS* yet.

In this paper, an extension of *GIFS*, called Liu' generalized intuitionistic fuzzy set (denoted by *L-GIFS*), is proposed, by good luck, if the extensional constant of *L-GIFS* $\tilde{A}$ equals to one, then the addition operation and scalar multiplication operation of Atanassov's *IFS* can still be used for this new *GIFS*, with the extensional index *L*=1, furthermore, an order-preserving partial order, called Liu's order, proposed by author's previous work (Liu, 2010) can still be used by this new *GIFV*, and it is pointed out that Atanassov's order is not a totally partial order, and Xu's order is not addition-invariant, both of them can not be used to construct the generalized intuitionistic fuzzy aggregation operators, on the contrary, Liu's order (Liu, 2010) is not only a totally partial order but also an addition-invariant one, which is order-preserving for addition operation and can be used for constructing an ordered semi-vector space of generalized intuitionistic fuzzy values to handle intuitionistic fuzzy multi-criteria fuzzy decision making problems.

II、Intuitionistic Fuzzy Set

The fundamental characteristic of a fuzzy set (Zadeh, 1965) is that it assigns to each of its element a membership degree and a non-membership degree under the constraint that the sum of two degrees equals one. Atanassov (1986, 1999) extended Zadeh's fuzzy sets to intuitionistic fuzzy sets (*IFSs*), which assigns to each of their elements a membership degree and a non-membership degree, under the constraint that the sum of two degrees does not exceeds one. Some formal definitions about *IFSs* are defined as follows:

Definition 1. Intuitionistic fuzzy sets (*IFSs*) (Atanassov, 1986, 1999)

Given a finite set *X*, an Atanassov's intuitionistic fuzzy set (*A-IFS*), $\tilde{A}$, is defined as

$$\tilde{A} = \left\{ \left\langle x, \mu_A(x), \gamma_A(x) \right\rangle \mid x \in X \right\} \tag{1}$$

which assigns to each element *x* a membership degree $\mu_A(x)$ and a non-membership degree $\gamma_A(x)$, satisfying following conditions:

(i) $$0 \le \mu_A(x) + \gamma_A(x) \le 1, \forall x \in X \tag{2}$$

(ii) $$0 \le \mu_A(x), \gamma_A(x), \pi_A(x) \le 1, \forall x \in X \tag{3}$$

(iii) $$\pi_A(x) = 1 - \mu_A(x) - \gamma_A(x) \tag{4}$$

Where $\pi_A(x)$ is called the intuitionistic fuzzy index of element x in *A-IFS* $\tilde{A}$, the value denotes a measure of non-determinacy. Obviously, if $\pi_A(x)=0$, then the intuitionistic fuzzy set, $\tilde{A}$, is just a Zadeh's fuzzy set.

Let $FS_Z(X)$ be the collection of all Zadeh's fuzzy sets on *X*, $IFS_A(X)$, be the collection of all *A-IFSs* on *X*. Obviously, we have $FS_Z(X) \subset IFS_A(X)$ (5)

III、Generalized Intuitionistic Fuzzy Set

In definition 1, to replace the condition (2) with the condition (6), the generalized intuitionistic fuzzy sets were proposed by Mondal and Samanta (2002), denoted by *MS-GIFSs*.

Definition 2. Mondal and Samanta's generalized intuitionistic fuzzy sets (*MS-GIFSs*)

Given a finite set *X*, A Mondal & Samanta's generalized intuitionistic fuzzy set (*MS-GIFS*), $\tilde{A}$, is defined as $\tilde{A}=\left\{\left\langle x,\mu_A(x),\gamma_A(x)\right\rangle \mid x\in X\right\}$, which assigns to each element x a membership degree $\mu_A(x)$ and a non-membership degree $\gamma_A(x)$, satisfying following conditions:

(i) $$0 \le \mu_A(x) \wedge \gamma_A(x) \le 0.5, \forall x \in X \tag{6}$$

(ii) $$0 \le \mu_A(x), \gamma_A(x), \pi_A(x) \le 1, \forall x \in X \tag{7}$$

(iii) $$\pi_A(x) = 1 - \mu_A(x) - \gamma_A(x) \tag{8}$$

Where $\pi_A(x)$ is called the Mondal and Samanta's generalized intuitionistic fuzzy index of element *x*, in *MS-GIFS* $\tilde{A}$, the value denotes a measure of non-determinacy.

The collection of all *MS-GIFSs* is denoted by $GIFS_{MS}(X)$, if $\pi_A(x)=0$, then the Mondal and Samanta's intuitionistic fuzzy set, $\tilde{A}$, is just a Zadeh's fuzzy set, and we have the following theorem:

Theorem 1. $IFS_A(X) \subset GIFS_{MS}(X)$ (9)

Proof. $\forall \tilde{A}=\left\{\left\langle x,\mu_A(x),\gamma_A(x)\right\rangle \mid x\in X\right\} \in IFS_A(X)$

We have $0 \le \mu_A(x)+\gamma_A(x) \le 1, \forall x \in X$, then

$$\mu_A(x) \wedge \gamma_A(x) \le \frac{\mu_A(x)+\gamma_A(x)}{2} \le 0.5 \tag{10}$$

Therefore, $\tilde{A} \in GIFS_{MS}(X)$, and the proof is completed.

Example 1. Let $X=\{x_1,x_2,x_3\}$ and $\tilde{A}=\left\{\left\langle x_1,0.8,0.3\right\rangle,\left\langle x_2,0.4,0.9\right\rangle,\left\langle x_3,0.5,0.4\right\rangle\right\} \in GIFS_{MS}(X)$, but $\tilde{A} \notin IFS_A(X)$, in other words, $GIFS_{MS}(X) \not\subset IFS_A(X)$

Note that $FS_Z(X) \subset IFS_A(X) \subset GIFS_{MS}(X)$ (11)

but $GIFS_{MS}(X) \not\subset IFS_A(X) \not\subset FS_Z(X)$ (12)

IV、An Extensional Generalized Intuitionistic Fuzzy Set

In this paper, an extensional generalized intuitionistic fuzzy set, called Liu's generalized intuitionistic fuzzy set, is proposed as bellows.

Definition 3. Liu's generalized intuitionistic fuzzy sets (*L-IFSs*)

Given a finite set *X*, and a constant $L \in [0,1]$, a Liu's generalized intuitionistic fuzzy set (*L-GIFS*), $\tilde{A}$, is defined as $\tilde{A} = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in X\}$, which assigns to each element *x* a membership degree $\mu_A(x)$ and a non-membership degree $\gamma_A(x)$, satisfying following conditions:

(i) $$0 \le \mu_A(x) + \gamma_A(x) \le 1 + L, \forall x \in X \tag{13}$$

(ii) $$0 \le \mu_A(x), \gamma_A(x) \le 1, \forall x \in X \tag{14}$$

(iii) $$\pi_A(x) = 1 + L - \mu_A(x) - \gamma_A(x) \tag{15}$$

Where L is called the extensional index of *L-GIFS* $\tilde{A}$, $\pi_A(x)$ is called the generalized intuitionistic fuzzy index of element *x* in *L-GIFS* $\tilde{A}$, the value denotes a measure of non-determinacy.

The collection of all *L-GIFSs* is denoted by $GIFS_L(X)$.

Theorem 2. (i) $L = \pi_A(x) = 0 \Rightarrow GIFS_L(X) = FS_Z(X)$ (16)

(ii) $L = 0 \Rightarrow GIFS_L(X) = IFS_A(X)$ (17)

(iii) $L = 1 \Rightarrow IFS_A(X) \subset GIFS_{MS}(X) \subset GIFS_L(X)$ (18)

Example 2. If $L = 1$, let $X = \{x_1, x_2, x_3\}$ and
$\tilde{A} = \{\langle x_1, 0.8, 0.6\rangle, \langle x_2, 0.6, 0.9\rangle, \langle x_3, 0.9, 0.7\rangle\} \in GIFS_L(X)$, but $\tilde{A} \notin GIFS_{MS}(X)$, in other words, $GIFS_L(X) \not\subset GIFS_{MS}(X)$

Note that $FS_Z(X) \subset IFS_A(X) \subset GIFS_{MS}(X) \subset GIFS_L(X)$ (19)

$GIFS_L(X) \not\subset GIFS_{MS}(X) \not\subset IFS_A(X) \not\subset FS_Z(X)$ (20)

and, hereafter, if there is no word mentioned the value of *L* in $GIFS_L(X)$, then it means $L = 1$.

V、Basic Algebraic Operations On $GIFS_L(X)$

Definition 4. Let $\tilde{A} = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in X\}$,
$\ddot{B} = \{\langle x, \mu_B(x), \gamma_B(x)\rangle \mid x \in X\} \in GIFS_L(X)$, then inclusion, equality, complementation, union, intersection, $\tilde{0}$ and $\tilde{1}$ on $GIFS_L(X)$ are defined as follows:

(i) $$\tilde{A} \subseteq \tilde{B}, \Leftrightarrow \mu_A(x) \le \mu_B(x), \gamma_A(x) \ge \gamma_B(x), \forall x \in X \tag{21}$$

(ii) $$\tilde{A} = \tilde{B}, \Leftrightarrow \tilde{A} \subseteq \tilde{B}, \tilde{B} \subseteq \tilde{A} \tag{22}$$

(iii) $$\tilde{A}^c = \{\langle x, \gamma_A(x), \mu_A(x)\rangle \mid x \in X\} \tag{23}$$

(iv) $$\tilde{A} \cup \tilde{B} = \{\langle x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle \mid x \in X\} \tag{24}$$

(v) $$\tilde{A} \cap \tilde{B} = \{\langle x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x) \mid x \in X\rangle\} \tag{25}$$

(vi) $$\tilde{0} = \left\{ \left\langle x, \mu_{\tilde{0}}(x), \gamma_{\tilde{0}}(x) \right\rangle \mid x \in X \right\}, \mu_{\tilde{0}}(x) = 0, \gamma_{\tilde{0}}(x) = 1, \forall x \in X \tag{26}$$

(vii) $$\tilde{1} = \left\{ \left\langle x, \mu_{\tilde{1}}(x), \gamma_{\tilde{1}}(x) \right\rangle \mid x \in X \right\}, \mu_{\tilde{1}}(x) = 1, \gamma_{\tilde{1}}(x) = 0, \forall x \in X \tag{27}$$

Definition 5. Let $\tilde{A} = \left\{ \left\langle x, \mu_A(x), \gamma_A(x) \right\rangle \mid x \in X \right\}$,
$\underset{\cdot\cdot}{B} = \left\{ \left\langle x, \mu_B(x), \gamma_B(x) \right\rangle \mid x \in X \right\} \in GIFS_L(X)$, then partial order, addition operation and scalar multiplication operation are defined as follows:

(i) $$\tilde{A} \le \tilde{B}, \Leftrightarrow \mu_A(x) < \mu_B(x), or\, \mu_A(x) = \mu_B(x) \gamma_A(x) \ge \gamma_B(x), \forall x \in X \tag{28}$$

(ii) $$\tilde{A} = \tilde{B}, \Leftrightarrow \tilde{A} \le \tilde{B}, \tilde{B} \le \tilde{A} \tag{29}$$

(iii) $$\tilde{A} \oplus \tilde{B} = \tilde{A} = \left\{ \left\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \gamma_A(x) \cdot \gamma_B(x) \right\rangle \mid x \in X \right\} \tag{30}$$

(iv) $$\alpha \tilde{A} = \begin{cases} \left\{ \left\langle x, 1 - [1 - \mu_A(x)]^{\alpha}, [\gamma_A(x)]^{\alpha} \right\rangle \mid x \in X \right\} & if\, \alpha > 0 \\ \tilde{0} & if\, \alpha = 0 \end{cases} \tag{31}$$

We can obtain following properties;

Theorem 3. Let $\tilde{A}, \tilde{B}, \tilde{C} \in GIFS_L(X), \alpha, \beta \ge 0$,we have

(i) $$\tilde{A} \bigcup \tilde{B}, \tilde{A} \bigcap \tilde{B}, \tilde{A}^c, \tilde{0}, \tilde{1}, \tilde{A} \oplus \tilde{B}, \alpha \tilde{A} \in GIFS_L(X) \text{ (closure)} \tag{32}$$

(ii) $$\tilde{A} \subseteq \tilde{B}, \tilde{B} \subseteq \tilde{C} \Rightarrow \tilde{A} \subseteq \tilde{C} \text{ (transitivity)} \tag{33}$$

(iii) $$\tilde{0} \subseteq \tilde{A} \subseteq \tilde{1}, \forall \tilde{A} \in GIFS_L(X) \text{ (boundary)} \tag{34}$$

(iv) $$\tilde{0} \bigcup \tilde{A} = \tilde{A} \bigcup \tilde{0} = \tilde{A}, \tilde{1} \bigcap \tilde{A} = \tilde{A} \bigcap \tilde{1} = \tilde{A}, \forall \tilde{A} \in GIFS_L(X) \tag{35}$$

(v) $$\tilde{A} \bigcup \tilde{B} = \tilde{B} \bigcup \tilde{A}, \tilde{A} \bigcap \tilde{B} = \tilde{B} \bigcap \tilde{A} \text{ (commutativity)} \tag{36}$$

(vi) $$(\tilde{A} \bigcup \tilde{B}) \bigcup \tilde{C} = \tilde{A} \bigcup (\tilde{B} \bigcup \tilde{C}) \text{ (associativity)} \tag{37}$$

(vii) $$(\tilde{A} \bigcap \tilde{B}) \bigcap \tilde{C} = \tilde{A} \bigcap (\tilde{B} \bigcap \tilde{C}) \text{ (associativity)} \tag{38}$$

(viii) $$\tilde{A} \bigcup (\tilde{B} \bigcap \tilde{C}) = (\tilde{A} \bigcup \tilde{B}) \bigcap (\tilde{A} \bigcup \tilde{C}) \text{ (distributivity)} \tag{39}$$

(ix) $$\tilde{A} \bigcap (\tilde{B} \bigcup \tilde{C}) = (\tilde{A} \bigcap \tilde{B}) \bigcup (\tilde{A} \bigcap \tilde{C}) \text{ (distributivity)} \tag{40}$$

(x) $$\tilde{A} \bigcup (\tilde{A} \bigcap \tilde{B}) = \tilde{A} = \tilde{A} \bigcap (\tilde{A} \bigcup \tilde{B}) \text{ (absorption)} \tag{41}$$

(xi) $$\tilde{A} \bigcup \tilde{A} = \tilde{A} \bigcap \tilde{A} = \tilde{A} \text{ (idempotence)} \tag{42}$$

(xii) $$(\tilde{A}^c)^c = \tilde{A} \text{ (involution)} \tag{43}$$

(xiii) $$(\tilde{A} \bigcup \tilde{B})^c = (\tilde{A}^c \bigcap \tilde{B}^c), (\tilde{A} \bigcap \tilde{B})^c = (\tilde{A}^c \bigcup \tilde{B}^c) \text{ (De morgan Law)} \tag{44}$$

(ixx) $$\tilde{A} \oplus \tilde{B} = \tilde{B} \oplus \tilde{A} \text{ (commutativity)} \tag{45}$$

(xx) $$(\tilde{A} \oplus \tilde{B}) \oplus \tilde{C} = \tilde{A} \oplus (\tilde{B} \oplus \tilde{C}) \text{ (associativity)} \tag{46}$$

(xxi) $$(\alpha + \beta) \tilde{A} = \alpha \tilde{A} \oplus \beta \tilde{A} \text{ (distributivity)} \tag{47}$$

(xxii) $$\alpha (\tilde{A} \oplus \tilde{B}) = (\alpha \cdot \tilde{A}) \oplus (\alpha \cdot \tilde{B}) \text{ (distributivity)} \tag{48}$$

(xxiii) $$\tilde{A} \oplus \tilde{0} = \tilde{0} \oplus \tilde{A} = \tilde{A} \text{ (unit element)} \tag{49}$$

VI、Semi-Ring

A、Semi-Ring

1. Semi-group

Definition 5. Semi-group (Golan,1999; Howie, 1976)

Let X be an nonempty set, $(X;\oplus)$ is called a semi-group on X with binary operations $\oplus: X\times X\to X$, if it satisfies the following conditions:

(i) $\forall a,b\in X \Rightarrow a\oplus b\in X$ (closure) (50)

(ii) $\forall a,b,c\in X \Rightarrow a\oplus(b\oplus c)=(a\oplus b)\oplus c$ (associativity) (51)

Theorem 4. $(GIFS_L(X);\bigcup)$ and $(GIFS_L(X);\bigcap)$ are two semi-groups.

2. Monoids

Definition 6. Monoid (Golan,1999, Howie, 1976)

A semi-groupo $(X;\oplus,e)$ is called a monoid, if $(X;\oplus)$ is a semi-group satisfying the following condition:

$e\in X,\forall a\in X \Rightarrow e\oplus a=a\oplus e=a$ (unit element) (52)

Theorem 5. $(GIFS_L(X);\bigcup,\tilde{0})$ and $(GIFS_L(X);\bigcap,\tilde{1})$ are two monoids.

3. Semi-ring

Definition 7. Semi-ring (Golan,1999; Howie, 1976)

Let X be an nonempty set, $(X;\oplus,\otimes,e,e')$ is called a semi-ring on X if $(X;\oplus,e)$ and $(X;\oplus,e')$ are two monoids on X satisfying the following conditions:

(i) $\forall a,b\in X \Rightarrow a\oplus b=b\oplus a$ (commutative) (53)

(ii) $e\in X,\forall a\in X \Rightarrow a\oplus e=e\oplus a=a, a\otimes e=e\otimes a=e$ (54)

(iii) $e'\in X,\forall a\in X \Rightarrow a\otimes e'=e'\otimes a=a$ (closure) (55)

(iv) $\forall a,b,c\in X \Rightarrow a\otimes(b\oplus c)=(a\otimes b)\oplus(a\otimes c)$ (distribution) (56)

Theorem 6. $(GIFS_L(X);\bigcup,\bigcap,\tilde{0},\tilde{1})$ is a semi-ring.

B、Partial orders

Definition 8. Partial order, Poset (Davey & Priestley, 2002; Dietrich & Hoffman,2003)

Let X be an nonempty set, a partial order, $\le$, on X is a binary relation which satisfies the following conditions:

(i) $a\le a,\forall a\in X$ (reflexivity) (57)

(ii) $a\le b, b\le a,\forall a,b\in X \Rightarrow a=b$ (antisymmetry) (58)

(iii) $a\le b, b\le c,\forall a,b,c\in X \Rightarrow a\le c$ (transitivity) (59)

If $\le$ is a partial order on X, then $(X,\le)$ is called a partially ordered set, or poset.

Definition 9. Totally partial order (Davey & Priestley, 2002; Dietrich & Hoffman, 2003)

Let $(X,\leq)$ be a poset, the partial order, $\leq$, is called a totally partial order on X, if it satisfies the following condition:

$$\forall a,b \in X \Rightarrow a \leq b \text{, or } b \leq a \text{ (reflexivity)} \tag{60}$$

Note that the partial order $\leq$ proposed in this paper is a totally partial order. However, the above-mentioned inclusion order $\subseteq$ is not a totally partial order.

C、Lattices

Definition 10. (Davey & Priestley, 2002)

Let $(X,\leq)$ be a poset, if $\forall a,b \in X \Rightarrow a \vee b, a \wedge b \in X$ (61)

Then $(X,\leq)$ or $(X,\vee,\wedge)$ is called a lattice.

Where $a \vee b = \vee\{a,b\} = \sup\{a,b\}, a \wedge b = \wedge\{a,b\} = \inf\{a,b\}$ (62)

Theorem 7. $(GIFS_L(X);\subseteq)$ or $(GIFS_L(X);\bigcup,\bigcap)$ is a lattice.

Proof. $\forall \tilde{A},\tilde{B} \in C_L(X), \tilde{A} \bigcup \tilde{B} = \sup\{\tilde{A},\tilde{B}\}, \tilde{A} \bigcap \tilde{B} = \inf\{\tilde{A},\tilde{B}\}$

$\sup\{\tilde{A},\tilde{B}\}, \inf\{\tilde{A},\tilde{B}\} \in GIFS_L(X)$

Hence, $(GIFS_L(X);\subseteq)$ or $(GIFS_L(X);\bigcup,\bigcap)$ is a lattice.

Definition 11. (Davey & Priestley, 2002)

Let $(X,\vee,\wedge)$ be a lattice, if it satisfies following two distributivity laws, then it is called a distributive lattice.

(i) $$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \forall a,b,c \in X \tag{63}$$

(ii) $$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \forall a,b,c \in X \tag{64}$$

Theorem 8. $(GIFS_L(X);\bigcup,\bigcap)$ is a distributive lattice.

Definition 12. (Davey & Priestley, 2002)

Let $(X,\vee,\wedge)$ be a lattice, if it satisfies following two distributivity laws, then it is called a completely distributive lattice.

(i) $$a \vee (\mathop{\wedge}_{i=1}^{n} b_i) = \mathop{\wedge}_{i=1}^{n} (a \vee b_i), \forall a,b_i \in X, n \in N \tag{65}$$

(ii) $$a \wedge (\mathop{\vee}_{i=1}^{n} b_i) = \mathop{\vee}_{i=1}^{n} (a \wedge b_i), \forall a,b_i \in X, n \in N \tag{66}$$

Theorem 9. $(GIFS_L(X);\bigcup,\bigcap)$ is a completely distributive lattice.

Definition 13. (Davey & Priestley, 2002)

Let $(X,\vee,\wedge)$ be a lattice, if it satisfies following bounded condition, then it is called a bounded lattice.

$$\exists 0,1 \in X, \forall a \in X \ni 0 \leq a \leq 1 \tag{67}$$

Theorem 10. $(GIFS_L(X);\bigcup,\bigcap)$ is a bounded lattice.

Definition 14. (Davey & Priestley, 2002)

$(X,\vee,\wedge,c)$ is a soft algebra, if $(X,\vee,\wedge)$ is a bounded and distributive lattice satisfying following two conditions.

(i) $$\forall a \in X, (a^c)^c = a \tag{68}$$

(ii) $$\forall a,b \in X \Rightarrow (a \vee b)^c = a^c \wedge b^c, (a \wedge b)^c = a^c \vee b^c \tag{69}$$

Theorem 11. $(GIFS_L(X);\bigcup,\bigcap,c)$ is a soft algebra.

Definition 15. (Davey & Priestley, 2002)

$(X,\vee,\wedge,c)$ is a superior soft algebra, if $(X,\vee,\wedge,c)$ is a completely distributive lattice satisfying following dense property.

$$\forall a,b \in X, a < b, (a \le b, a \ne b) \Rightarrow \exists r \in X \ni a < r < b \tag{70}$$

Theorem 12. $(GIFS_L(X);\bigcup,\bigcap,c)$ is a superior soft algebra.

Proof. $\forall \tilde{A},\tilde{B} \in X, \tilde{A} < \tilde{B}$, that is $\tilde{A} \ne \tilde{B}, \tilde{A} \le \tilde{B}$

Let $\tilde{A} = \left\{\left\langle x, \mu_A(x), \gamma_A(x)\right\rangle \mid x \in X\right\}$, $\tilde{B} = \left\{\left\langle x, \mu_B(x), \gamma_B(x)\right\rangle \mid x \in X\right\}$

And $$\tilde{R} = \left\{\left\langle x, 0.5(\mu_A(x)+\mu_B(x)), 0.5(\gamma_A(x)+\gamma_B(x))\right\rangle \mid x \in X\right\} \tag{71}$$

then $\tilde{R} \in GIFS_L(X)$ and

(i) $$\mu_A(x) = \mu_B(x), \gamma_A(x) > \gamma_B(x) \text{ ,or} \tag{72}$$

(ii) $$\mu_A(x) < \mu_B(x), \gamma_A(x) = \gamma_B(x) \text{ ,or} \tag{73}$$

(iii) $$\mu_A(x) < \mu_B(x), \gamma_A(x) > \gamma_B(x) \tag{74}$$

we can obtain $\tilde{A} < \tilde{R} < \tilde{B}$.

Therefore $(GIFS_L(X);\bigcup,\bigcap,c)$ is a superior soft algebra.

Note that $(P(X);\bigcup,\bigcap,c)$ is not a superior soft algebra, where $P(X)$ is the power set of *X*.

VII、Intuitionistic Fuzzy Values (*IFVs*)

To be convenience, corresponding *IFS*, the intuitionistic fuzzy values(*IFVs*) is defined as follows (Xu, 2007):

Definition 16. Intuitionistic fuzzy values (*IFVs*) (Xu, 2007)

If $\mu,\gamma \ge 0$, and $(\mu+\gamma) \le 1$, then the order pair (μ,γ) is called an intuitionistic fuzzy value (*IFV*).

In this paper, corresponding *GIFS*, the generalized intuitionistic fuzzy values (*GIFVs*) is defined as follows:

Definition 17. generalized intuitionistic fuzzy values (*GIFVs*).

If $0 \le \mu,\gamma \le 1$, then the order pair (μ,γ) is called Liu's generalized intuitionistic fuzzy value $(GIFV_L)$.

Corresponding the operations on *IFS* and *GIFS*, we can obtain following two useful operations on *IFV* and $GIFV_L$.

Definition 18. Two useful operations on *IFV* (Atanassov, 1986, 1999; De, Biswas & Roy, 2000)

Let $S(IFV)$ be the set of all *IFVs*, and $S(GIFV_L)$ be the set of all $GIFV_L s$. If $\tilde{a}=(\mu_a,\gamma_a),\tilde{b}=(\mu_b,\gamma_b)\in S(GIFV_L)$, then

(i) $$\tilde{a}\oplus\tilde{b}=(\mu_a+\mu_b-\mu_a\mu_b,\gamma_a\gamma_b) \tag{75}$$

(ii) $$\lambda\cdot\tilde{a}=\begin{Bmatrix}(1-(1-\mu_a)^{\lambda},(\gamma_a)^{\lambda}) & \lambda>0\\ (0,1) & \lambda=0\end{Bmatrix} \tag{76}$$

Furthermore, we can get the following theorem:

Theorem 13. If $\tilde{a},\tilde{b},\tilde{c}\in P,\alpha,\beta>0$ then

(i) $$\tilde{a}\oplus\tilde{b},\lambda\tilde{a},(0,1)\in S(GIFV_L) \tag{77}$$

(ii) $$\tilde{a}\oplus\tilde{b}=\tilde{b}\oplus\tilde{a} \tag{78}$$

(iii) $$\tilde{a}\oplus(\tilde{b}\oplus\tilde{c})=(\tilde{a}\oplus\tilde{b})\oplus\tilde{c} \tag{79}$$

(iv) $$(0,1)\oplus\tilde{a}=\tilde{a}\oplus(0,1)=\tilde{a} \tag{80}$$

(v) $$\forall\tilde{a}\in X,\forall\alpha>0\Rightarrow\alpha\cdot\tilde{a}=\tilde{a}\cdot\alpha \tag{81}$$

(vi) $$\forall\tilde{a}\in X,\forall\alpha,\beta>0\Rightarrow(\alpha+\beta)\tilde{a}=\alpha\tilde{a}\oplus\beta\tilde{a} \tag{82}$$

(vii) $$\forall\alpha\in R^{+},\tilde{a},\tilde{b}\in X\Rightarrow\alpha\cdot(\tilde{a}\oplus\tilde{b})=(\alpha\cdot\tilde{a})\oplus(\alpha\cdot\tilde{b}) \tag{83}$$

VIII、Ordered Semi-Vector Space

A、Addition-invariant partial orders

Definition 19. Addition-invariant partial orders (Dietrich & Hoffman, 2003; Liu, 2010)

(i) Let *X* be an nonempty set with an addition binary operations $\oplus: X\times X\to X$ and a totally partial order, $\le$, on *X*, if it satisfies the following condition, then the algebraic structure $(X;\oplus,\le)$ is called an addition order-preserving structure,

$$\forall a,b,c\in X, a\le b\Rightarrow a\oplus c\le b\oplus c \tag{84}$$

(ii) If $(X;\oplus,\le)$ is an addition order-preserving structure , then the totally partial order , $\le$, is called an addition-invariant partial order.

B、Three kinds of partial orders

Definition 20. Atanassov's order (Atanassov, 1986, 1999)

Let $S(IFV)$ be the set of all *IFVs*, $\tilde{a}=(\mu_a,\gamma_a),\tilde{b}=(\mu_b,\gamma_b)\in S(IFV)$ (85)

Atanassov's order , $\le_A$, is a binary relation which satisfies the following conditions:

$$\tilde{a}\le_A\tilde{b}\Leftrightarrow\mu_a\le\mu_b,\gamma_a\ge\gamma_b \tag{86}$$

Note that Atanassov's order is a partial order, but not a totally partial order, since if $\mu_a<\mu_b,\gamma_a<\gamma_b$ or $\mu_a>\mu_b,\gamma_a>\gamma_b$, then $\tilde{a}$ and $\tilde{b}$ are incomparable. It can be only used for *IFS* and *GIFS*, not for *IFV* and *IFV*.

Definition 21. Xu's order (Xu, 2007)

Let $S(IFV)$ be the set of all $IFVs$, $\tilde{a}=(\mu_a,\gamma_a),\tilde{b}=(\mu_b,\gamma_b)\in P$

$$S(\tilde{a})=(\mu_a-\gamma_a),H(\tilde{a})=(\mu_a+\gamma_a) \quad (87)$$

Xu's partial order, $\leq_X$, is defined as follows:

(i) $$\tilde{a}<_X\tilde{b}\Leftrightarrow S(\tilde{a})<S(\tilde{b}), or S(\tilde{a})=S(\tilde{b}),H(\tilde{a})<H(\tilde{b}) \quad (88)$$

(ii) $$\tilde{a}=\tilde{b}\Leftrightarrow S(\tilde{a})=S(\tilde{b}),H(\tilde{a})=H(\tilde{b}) \quad (89)$$

(iii) $$\tilde{a}\leq_X\tilde{b}\Leftrightarrow \tilde{a}<_X\tilde{b}, or \tilde{a}=\tilde{b} \quad (90)$$

Definition 22. Liu's order (Liu, 2010)

Let $S(GIFV_L)$ be the set of all $GIFV_L s$, $\tilde{a}=(\mu_a,\gamma_a),\tilde{b}=(\mu_b,\gamma_b)\in P$, Liu's order, $\leq_L$, is defined as follows:

(i) $$\tilde{a}<_L\tilde{b}\Leftrightarrow \mu_a<\mu_b, or \mu_a=\mu_b,\gamma_a>\gamma_b \quad (91)$$

(ii) $$\tilde{a}=\tilde{b}\Leftrightarrow \mu_a=\mu_b,\gamma_a=\gamma_b \quad (92)$$

(iii) $$\tilde{a}\leq_L\tilde{b}\Leftrightarrow \tilde{a}=\tilde{b}, or \tilde{a}<_L\tilde{b} \quad (93)$$

C、Semi-vector space

Definition 23. Semi-vector space (Pap, 1979)

Let *X* be an nonempty set, $(X;\oplus,\cdot)$ is called a semi-vector space on *X* over R^+ with binary operations $\oplus: X\times X\to X$, and $\cdot: R^+\times X\to X$, if it satisfies the following conditions:

(i) $(X;\oplus)$ is a semi-group

(ii) $\forall a,b\in X\Rightarrow a\oplus b=b\oplus a$ (commutative) (94)

(iii) $\forall\lambda\in R^+, a\in X\Rightarrow \lambda\cdot a\in X$ (closure) (95)

(iv) $\forall a\in X,\forall\alpha\in R^+\Rightarrow \alpha\cdot a=a\cdot\alpha$ (commutative) (96)

(v) $\forall\alpha,\beta\in R^+, a\in X\Rightarrow (\alpha\beta)\cdot a=\alpha(\beta\cdot a)$ (association) (97)

(vi) $\forall\alpha\in R^+, a,b\in X\Rightarrow \alpha\cdot(a\oplus b)=(\alpha\cdot a)\oplus(\alpha\cdot b)$ (distribution) (98)

(vii) $\forall\alpha,\beta\in R^+, a\in X\Rightarrow (\alpha+\beta)\cdot a=(\alpha\cdot a)\oplus(\beta\cdot a)$ (distribution) (99)

(viii) $1\in R^+, a\in X\Rightarrow 1\cdot a=a\in X$ (unit) (100)

Definition 24. Completed semi-vector space (Pap, 1979)

Let $(X;\oplus,\cdot)$ be a semi-vector space on X over R^+, if $\exists e\in X,\forall a\in X\Rightarrow e\oplus a=a\oplus e=a$, then $(X;\oplus,\cdot)$ is called a completed semi-vector space.

Theorem 14. Let be the set of all *IFVs* $(S(GIFV_L);\oplus,\cdot)$ be a semi-vector space on $S(GIFV_L)$ over R^+ "$\oplus$" and "$\cdot$" be the operations on $S(GIFV_L)$ defined as Eqs. (75) and (76), then $(S(GIFV_L);\oplus)$ is a monoid.

Proof. It is trivial by Theorem 1.

D、Ordered semi-vector space, operation-invariant

Definition 25. Ordered semi-vector space

A semi-vector space on X over R^{+} $(X;\oplus,\cdot)$ with a totally partial order, $\leq$, on X over R^{+}, denoted by $(X;\oplus,\cdot,\leq)$, is called an ordered semi-vector space, if it satisfies the following order-preserving condition:

$$\forall a,b,c\in X, a\leq b\Rightarrow a\oplus c\leq b\oplus c \tag{101}$$

Definition 26. Operations-invariant (Liu, 2010)

If $(X;\oplus,\cdot,\leq)$, is an ordered semi-vector space, we called that the partial order $\leq$ is operations-invariant.

Note that. Let $\oplus$ be the operations on $S(GIFV_L)$ defined as Eq. (75), $(S(GIFV_L);\oplus,\cdot)$ be a semi-vector space and $\leq_X$ be Xu's partial order, then $(S(GIFV_L);\oplus,\cdot,\leq_X)$ is not an odered semi-vector space, in other words, $\leq_X$ is not operations-invariant.
Here, a counterexample is given as bellows.

Example 3. Counterexample

(i) Let $\tilde{a},\tilde{b},\tilde{c}\in S(IFV)\subset S(GIFV_L)$

$$\tilde{a}=(\mu_a,\gamma_a)=(0.5,0.3),\tilde{b}=(\mu_b,\gamma_b)=(0.4,0.1)\in P \tag{102}$$

$$\tilde{c}=(\mu_c,\gamma_c)=(0.1,0.1),\lambda=2 \tag{103}$$

Then $S(\tilde{a})=\mu_a-\gamma_a=0.5-0.3=0.2$ (104)

$$S(\tilde{b})=\mu_b-\gamma_b=0.4-0.1=0.3 \tag{105}$$

Since $S(\tilde{a})=0.2<S(\tilde{b})=0.3$ then $\tilde{a}<_X\tilde{b}$

(ii) From (i) we can obtain

$$\tilde{a}\oplus\tilde{c}=(\mu_a+\mu_c-\mu_a\mu_c,\gamma_a\gamma_c)=(0.55,0.03) \tag{106}$$

$$\tilde{b}\oplus\tilde{c}=(\mu_b+\mu_c-\mu_b\mu_c,\gamma_b\gamma_c)=(0.46,0.01) \tag{107}$$

and $S(\tilde{a}\oplus\tilde{c})=0.55-0.03=0.52>S(\tilde{b}\oplus\tilde{c})=0.46-0.01=0.45$ (108)

By Xu's order, we have $\tilde{a}\oplus\tilde{c}>_X\tilde{b}\oplus\tilde{c}$.
Therefore Xu's order is not addition invariant.

Theorem 15. Let $\oplus$ and $\cdot$ be the operations on $S(GIFV_L)$ over R^{+} defined as Eqs (75), $(S(GIFV_L);\oplus,\cdot)$ be a semi-vector space on X over R^{+} and $\leq_L$ be Liu's order, then $(S(GIFV_L);\oplus,\cdot,\leq_L)$ is an ordered semi-vector space.

Proof.

Let $\tilde{a}=(\mu_a,\gamma_a),\tilde{b}=(\mu_b,\gamma_b),\tilde{c}=(\mu_c,\gamma_c)\in S(GIFV_L)$ (109)
and $\lambda>0$
To prove that $\leq_L$ is addition-invariant,
If $\tilde{a}\leq\tilde{b}$, then

$$\begin{aligned}\tilde{a}\oplus\tilde{c}&=(\mu_a+\mu_c-\mu_a\mu_c,\gamma_a\gamma_c)\\ \tilde{b}\oplus\tilde{c}&=(\mu_b+\mu_c-\mu_b\mu_c,\gamma_b\gamma_c)\end{aligned} \tag{110}$$

If $\tilde{a} = \tilde{b} \Rightarrow \tilde{a} \oplus \tilde{c} = \tilde{b} \oplus \tilde{c} \Rightarrow \tilde{a} \oplus \tilde{c} \leq \tilde{b} \oplus \tilde{c}$ (111)

If $\tilde{a} \leq_L \tilde{b} \Rightarrow \mu_a < \mu_b$, or $\mu_a = \mu_b, \gamma_a > \gamma_b$ (112)

(i) If $\mu_a < \mu_b$, we can obtain

$$\begin{aligned} &\mu_a(1-\mu_c) \leq \mu_b(1-\mu_c) \\ &\Rightarrow \mu_a + \mu_c - \mu_a\mu_c < \mu_b + \mu_c - \mu_b\mu_c \\ &\Rightarrow \tilde{a} \oplus \tilde{c} \leq_L \tilde{b} \oplus \tilde{c} \end{aligned} \tag{113}$$

(ii) If $\mu_a = \mu_b, \gamma_a > \gamma_b$, we can obtain

$$(\mu_a + \mu_c - \mu_a\mu_c, \gamma_a\gamma_c) = (\mu_b + \mu_c - \mu_b\mu_c, \gamma_b\gamma_c) \tag{114}$$

and $\gamma_a\gamma_c \geq \gamma_b\gamma_c)$, then $\tilde{a} \oplus \tilde{c} \leq_L \tilde{b} \oplus \tilde{c}$ (115)

The proof is completed.

IX、Conclusions

In this paper, an extensional generalized intuitionistic fuzzy set, called Liu's generalized intuitionistic fuzzy set (*L-GIFS*) is proposed. It is showed that not only Atanassov's intuitionistic fuzzy set but also Mondal and Samanta's generalized intuitionistic fuzzy set is a special case of this new one. Responding the *L-GIFS*, the Liu's generalized intuitionistic fuzzy value $(GIFV_L)$ is also proposed. Some comparable examples are given, and some important notions and basic algebraic properties of *L-GIFSs* and $GIFV_L s$ are discussed.

Moreover, it is proved that $GIFV_L(X)$ is a semi-ring, completely distributive lattice and a superior soft algebra, and $S(GIFV_L)$ is a semi-vector space.

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